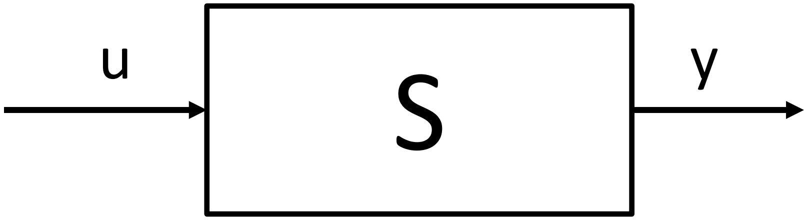
Lecture #0.1

Math tools for Control Engineering

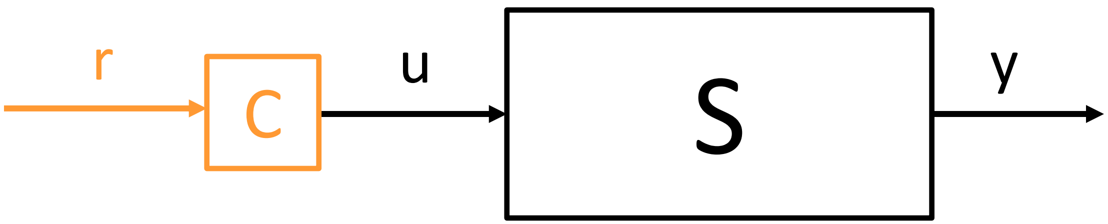
Overview

- Introduction to dynamical systems
- LTI systems
- Time domain response and Lagrange formula
- Laplace transform and transfer function
- Equilibria and stability
- Example: state feedback
- Harmonic response function
- Discrete-time systems (hints)

Introduction



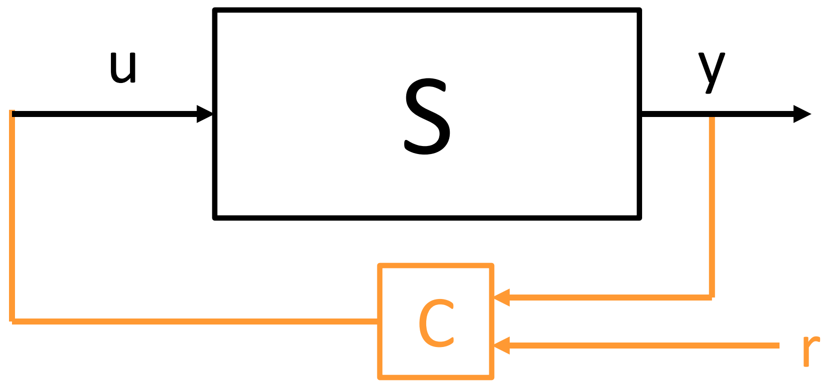
- We want to assign a desired behaviour to the system S
- the output $y(t)$ should be close to some reference signal $r(t)$
- To do so, we look for a suitable input $u(t)$



- Two possible strategies

1. *Open-loop*: find u that allows to reach the desired y *a priori* (usually based on an **inverse model** of the system) (*feedforward* control)

Introduction



- Two possible strategies

2. ***Closed-loop***: calculate u ***online*** based on a **measurement of the output** y (*feedback* control)

Open-loop control

Example:

$$c\dot{T} = -k(T - T_e) + P_{in}$$

- T oven temperature [°C]
- T_e external temperature [°C]
- c thermal capacity [J/°C]
- k thermal conductivity [W/°C]
- P_{in} input power [W]

Open-loop control

Example: $c\dot{T} = -k(T - T_e) + P_{in}$

- Desired steady-state temperature of 150°, assuming

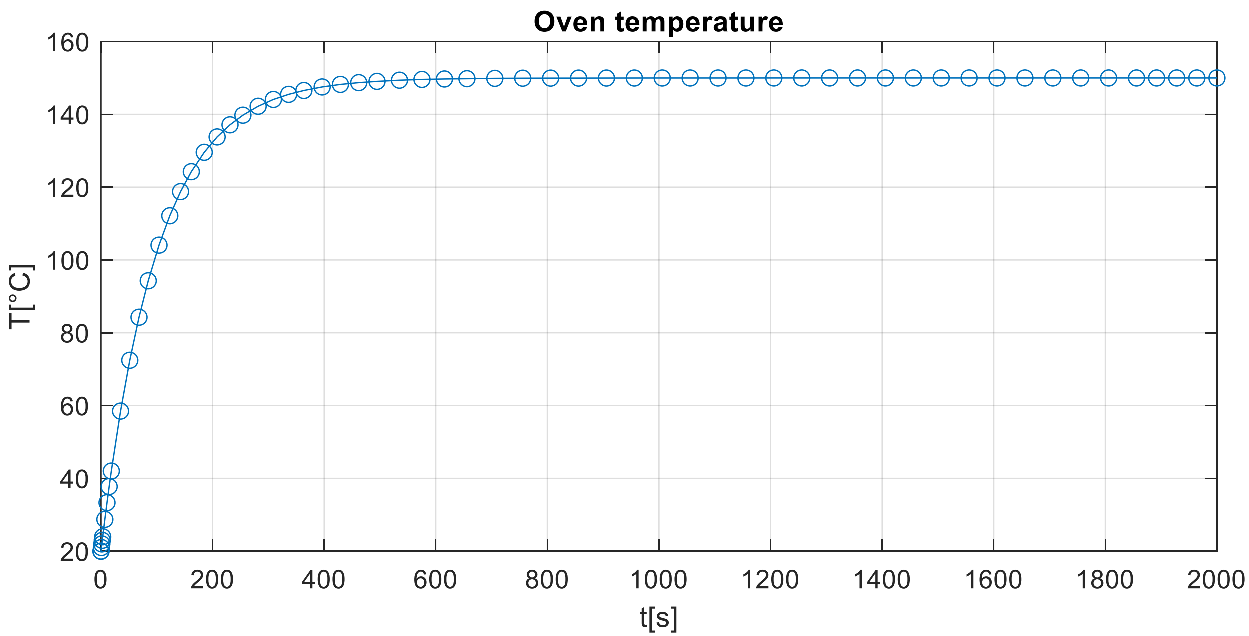
$$T_e = 20$$
$$\mathrm{k} = 10$$
$$c = 1000$$

→ feedforward strategy:

$$P_{in} = k(150° - T_e) = 1300 \text{ W}$$

Open-loop control

```
>> T_e = 20;
>> c = 1000;
>> k = 10;
>> ode45(@(t,T)-k/c*(T-T_e)+P_in/c,[0 2000],T_e)
```



Open loop control

- What could go wrong?
  - uncertainty on model coefficients
  - uncertainty on environment temperature
  - not all power levels may be available
    → relay/bang-bang control
  - may be *slow*…
    → improve dynamical performance
- Moreover, feedforward is not viable for unstable systems!

Why feedback control?

Why design a feedback controller?

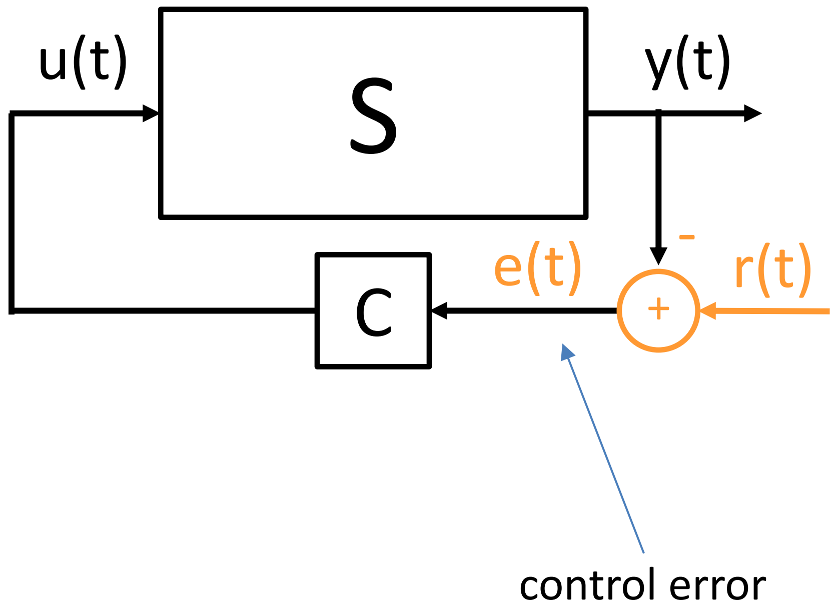
1. **Uncertainties:**
   the model may not be accurate
2. **External disturbances:**
   unforeseen and uncontrollable inputs may act on the system
3. **Efficiency:**
   optimize the time/amount of energy needed
4. **Flexibility:**
   we may not know the reference in advance

Why feedback control?

- But above all:
  - open-loop control changes the **output** of the system
  - closed-loop control **changes system dynamics**:
    → we can stabilize an unstable system!

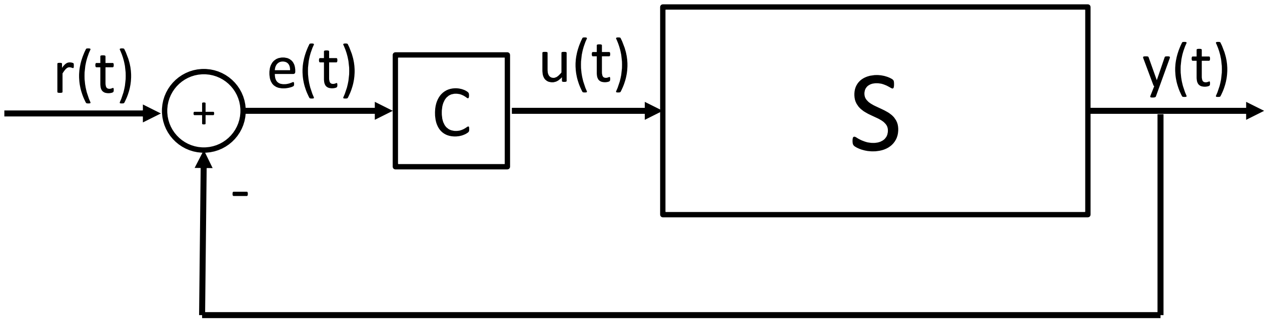
Closed-loop control

- The reference is usually **compared with the measured output** to generate a **control error signal**



Closed-loop control

Classic closed-loop control scheme



- r(t): reference
- e(t): control error
- u(t): control action
- y(t): controlled output

Dynamical systems

- Dynamical system : mathematical model that represents how a given process (**system**) with a certain number of degrees of freedom evolves over time

- Behavior typically depends on **previous history**, which is compactly captured by a set of state variables
(e.g. position and speed of a material point)

- Typically, evolution is described by a set of differential equations

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Dynamical systems

- General equations of a dynamical system

$$\dot{\boldsymbol{x}}(t) = \boldsymbol{f}(\boldsymbol{x}(t), \boldsymbol{u}(t); t), \qquad x(t_0) = x_0$$
$$\underbrace{\boldsymbol{y}(t)}_{\text{outputs}} = \boldsymbol{g}(\underbrace{\boldsymbol{x}(t)}_{\text{state}}, \underbrace{\boldsymbol{u}(t)}_{\text{inputs (controllable or not)}}; t)$$

- This representation is called **Input-State-Output** (or **ISO**)
- In general, a system can be non-linear, time-varying, MIMO, complex, multi-agent…

LTI dynamical systems

- Special case: Linear Time-Invariant (SISO) systems

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + Bu(t), \qquad x(t_0) = x_0$$
$$y(t) = C\boldsymbol{x}(t) + Du(t)$$

- linear systems → superposition principle!

Free evolution

- We take advantage of the superposition principle to decompose the system as (#1)

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}, \qquad \boldsymbol{x}(t_0) = \boldsymbol{x}_{\mathbf{0}}$$

$$\boldsymbol{u} = \mathbf{0}$$

<u>NOTE</u>: a system whose state **does not depend on external inputs** is said to be autonomous (but the same terminology is also used for nonlinear systems that do not depend explicitly on time)

Free evolution

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}$$

- This system evolves according to

$$\boldsymbol{x}(t) = e^{A(t-t_0)}\boldsymbol{x_0}$$

Matrix exponential
$$\frac{d}{dt}e^{At} = Ae^{At}$$

→ free evolution/natural response $(u = 0, \;\; x(t_0) \neq 0)$

NOTE: the dependency on $t - t_0$ and not on t and t_0 separately comes from the **time-invariance hypothesis**

Forced response

- We take advantage of the superposition principle to decompose the system as (#2)

$$\dot{x} = Ax + Bu, \qquad x(t_0) = 0$$

$$x(t) = \int_{t_0}^{t} e^{A(t-\tau)} B u(\tau) d\tau$$

→ forced response $(u \neq 0, \; x(t_0) = 0)$

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Lagrange formula

- Adding up the two contributions

$$\dot{x} = Ax + Bu, \qquad x(t_0) = x_0$$
$$y = Cx + Du$$

Convolution integral also written as $\boldsymbol{e^{At} \star u(t)}$

$$x(t) = e^{A(t-t_0)}x_0 + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$
$$y(t) = Cx(t) + Du(t)$$

Lagrange's formula

Lagrange formula

- Assume for simplicity D = 0* (strictly proper system)
  *If not, the system could be separated into a dynamical and a purely algebraic part (D)

$$\dot{x} = Ax + Bu, \qquad x(t_0) = x_0$$
$$y = Cx + \cancel{D}u$$

$$\Downarrow$$

$$y(t) = Ce^{A(t-t_0)}x_0 + C\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

NOTE: from here on we will assume (wlog) $t_0 = 0$

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Natural modes

- We defined the **matrix exponential function** through the property

$$\frac{d}{dt}e^{At} = Ae^{At}$$

- But we can also express it through its **Taylor series**

$$e^{At} = I + \sum_{k} \frac{A^k}{k!} t^k$$

Natural modes

- In the case of a **diagonal matrix**

$$\Lambda = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \lambda_n \end{bmatrix} \Rightarrow e^{\Lambda} = I + \sum_k \frac{\Lambda^k}{k!} = \begin{bmatrix} e^{\lambda_1} & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & e^{\lambda_n} \end{bmatrix}$$

- More in general, if the dynamical matrix is diagonalizable*:

$$\Lambda = T^{-1}AT$$
$$\dot{\boldsymbol{x}} = T\Lambda T^{-1}\boldsymbol{x}$$
$$\left(T^{-1}\dot{\boldsymbol{x}}\right) = \Lambda\left(T^{-1}\boldsymbol{x}\right)$$

* For simplicity we only consider the case of eigenvalues with multiplicity one

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Natural modes

Defining

$$\boldsymbol{\xi} = T^{-1}\boldsymbol{x}$$

we get

$$\begin{aligned}\boldsymbol{\xi}(t) &= e^{\Lambda t}\boldsymbol{\xi}_{\mathbf{0}} = \\ &= \xi_{0,1}e^{\lambda_1 t} + \xi_{0,2}e^{\lambda_2 t} + \cdots + \xi_{0,n}e^{\lambda_n t}\end{aligned}$$

$$\boldsymbol{x}(t) = T\boldsymbol{\xi}(t) = \underbrace{Te^{\Lambda t}T^{-1}}_{= e^{At}}\boldsymbol{x}_{\mathbf{0}}$$

The evolution of the system is associated with exponential modes related to the eigenvalues of the A matrix

Natural modes

These modes can be:

- Aperiodic → real eigenvalues

$$c_i e^{\lambda_i t}$$

- Pseudoperiodic → complex eigenvalues

$$c_i e^{\alpha_i t} e^{i\omega_i t}$$

NOTE: in the case of complex eigenvalues, a pseudoperiodic mode will correspond to a **pair** of conjugated eigenvalues. More details here

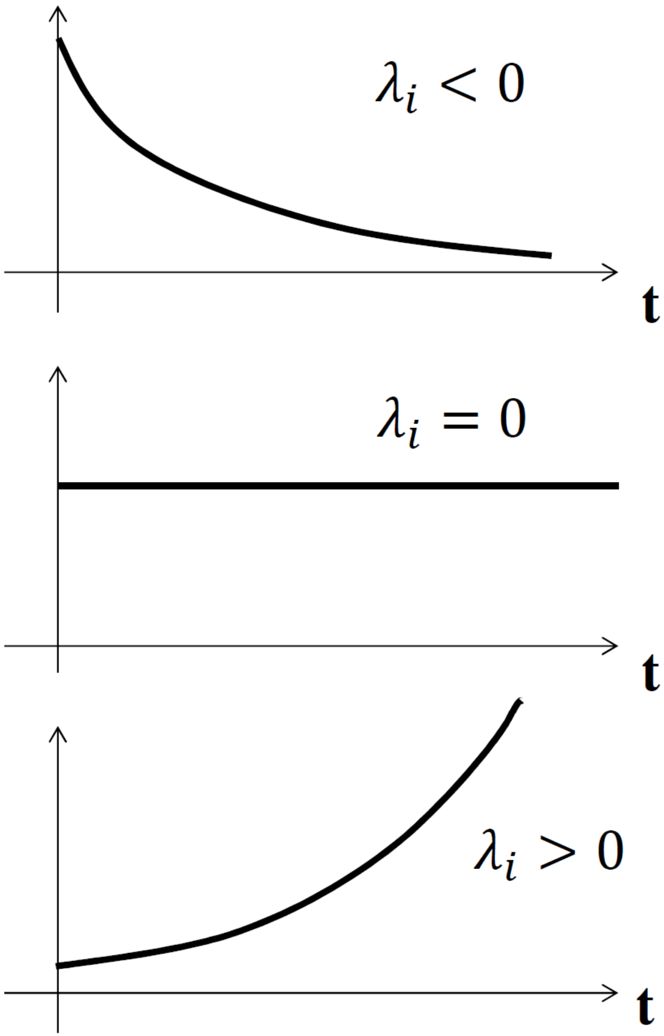
Natural modes

Aperiodic modes

– Convergent

– Constant

– Divergent



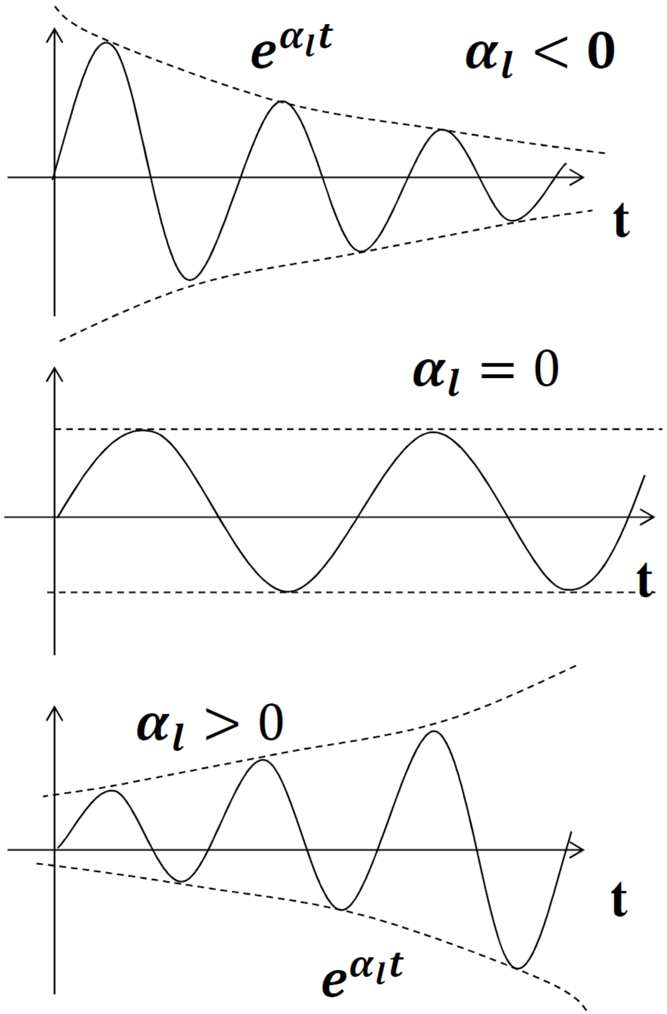
Natural modes

Pseudo-periodic modes

– Convergent

– Constant

– Divergent



Hurwitz theorem

- A is said to be a **Hurwitz** (or **stable**) **matrix** if all of its eigenvalues have **negative real part**

- Dynamical systems with a Hurwitz A matrix are **stable**

Laplace Transform

- We can apply the Laplace Transform to the equations of our system

$$L[f(t)] = F(s) \coloneqq \int_0^{\infty} f(t)e^{-st}dt$$

- We consider the **monolateral** transform: we want to preserve **causality**
- Mono and bi-lateral transforms coincide if $f(t) = f(t) \cdot \mathbb{1}(t)$

Laplace Transform

Laplace transform has the following properties

- Linearity
- Translation in the Laplace domain

$$L[e^{\alpha \mathrm{t}} f(t)] = F(s-\alpha)$$

- Translation in time

$$L[f(t-T)] = F(s)e^{-sT}$$

Laplace Transform

Laplace transform has the following properties

- Derivative in time

$$L\left[\frac{df}{dt}\right] = sF(s) - f(0)$$

- Integral in time

$$L\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}F(s)$$

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Laplace Transform

Laplace transform has the following properties

- Convolution

$$L[f * g] = F(s)G(s)$$

→ thanks to this property, we can turn the convolution integral seen before into a product of Laplace transforms (see later)

Laplace Transform

The following theorems also apply

- Initial value

$$f(0) = \lim_{s \to \infty} sF(s)$$

- Final value

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$$

List of transforms

- Dirac's delta fcn
$$L[\delta(t)] = 1$$
- Step fcn
$$L[\mathbb{1}(t)] = 1/s$$
- Ramp
$$L[R(t)] = 1/s^2$$
- Exponential
$$L[e^{\alpha t}] = \frac{1}{s-\alpha}$$
- Sine
$$L[\sin(\omega t)] = \frac{\omega}{s^2+\omega^2}$$
- Cosine
$$L[\cos(\omega t)] = \frac{s}{s^2+\omega^2}$$

Transfer function

- We can apply the Laplace Transform to the equations of our system

$$sX(s) - x_0 = AX(s) + BU(s)$$
$$Y(s) = CX(s) + DU(s)$$

⬇

$$Y(s) = C(sI - A)^{-1}x_0 + \underbrace{\left(C(sI - A)^{-1}B + D\right)}_{\text{transfer function}}U(s)$$

transfer function

Transfer function

The expression

$$H(s) := \frac{Y(s)}{U(s)} = C(sI - A)^{-1}B$$

is also called **transfer function** of the system

→ The tf is an **Input-Output (IO)** representation of the system

Transfer function

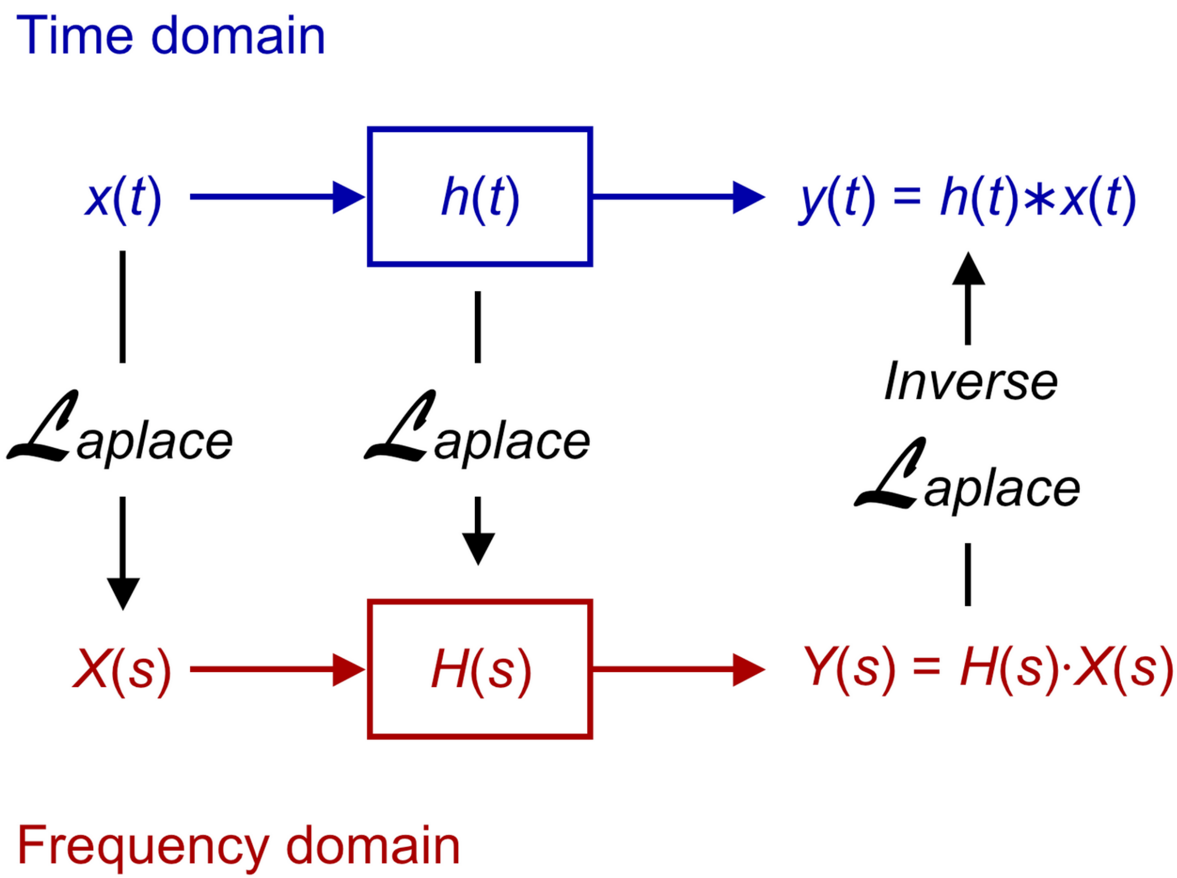
In particular, since $L[\delta(t)] = 1$

$$\mathrm{u(t)} = \delta(t) \Rightarrow \mathrm{Y(s)} = H(s)U(s) = H(s)$$

→ The transfer function of a linear system is the Laplace transform of the impulse response

Transfer function

Real differential equation → Complex algebraic equation



Transfer function

For LTI systems, the tf will be a **ratio of polynomials**

$$H(s) = \frac{num(s)}{den(s)} = \frac{a_m s^m + a_{m-1} s^{m-1} + \cdots + a_1 s + a_0}{b_n s^n + b_{n-1} s^{n-1} + \cdots + b_1 s + b_0}$$

- The roots of num(s) are called **zeros**
- The roots of den(s) are called **poles**
- $den(s) = \det(sI - A)$ → **characteristic polynomial** of A!

The poles coincide with the eigenvalues of the system

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Transfer function

The tf can be broken down to terms of the form

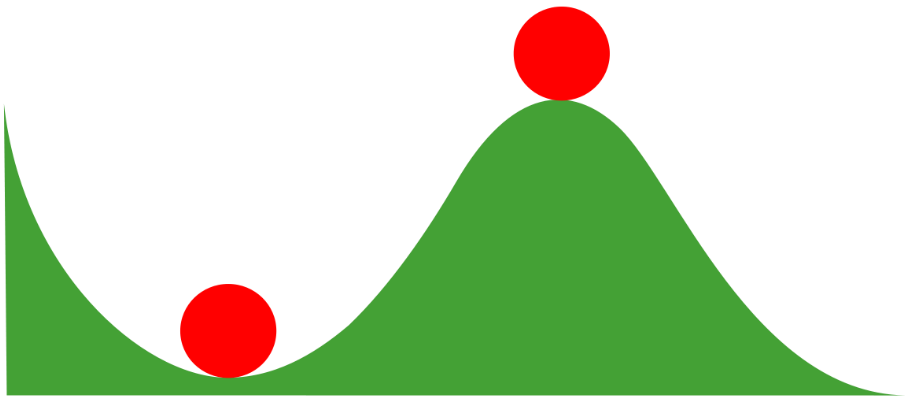
- $\frac{1}{s-\lambda} \xrightarrow{L^{-1}} e^{\lambda t} \cdot \mathbb{1}(\mathrm{t})$
- $\frac{\omega}{(s-\alpha)^2+\omega^2} \xrightarrow{L^{-1}} e^{\alpha t}\sin(\omega t) \cdot \mathbb{1}(\mathrm{t})$
- $\frac{s}{(s-\alpha)^2+\omega^2} \xrightarrow{L^{-1}} e^{\alpha t}\cos(\omega t) \cdot \mathbb{1}(\mathrm{t})$

→ Again the natural evolution modes seen before!

Equilibria and stability

- Equilibrium points are defined by the equation

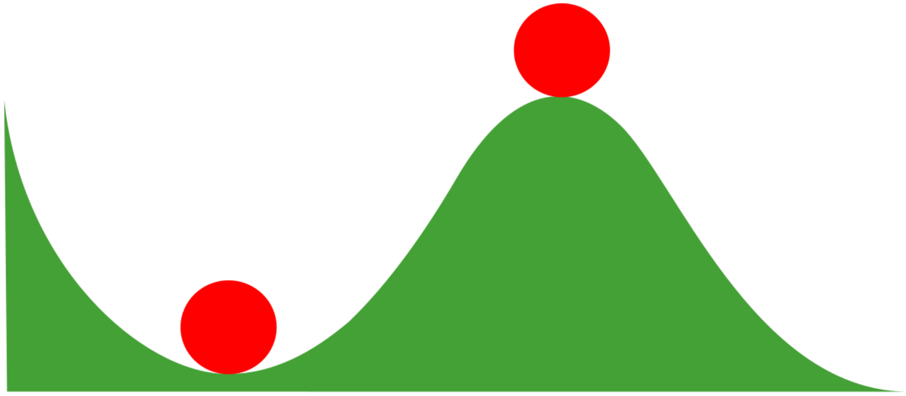
$$\dot{x} = f(x, u) = 0$$



Equilibria and stability

- Without external forcing, a system that is at an equilibrium point will stay there

- However, equilibria can be **stable or unstable**



Equilibria and stability

- For linear systems, the only equilibrium point is

$$\boldsymbol{Ax} = \boldsymbol{0}$$

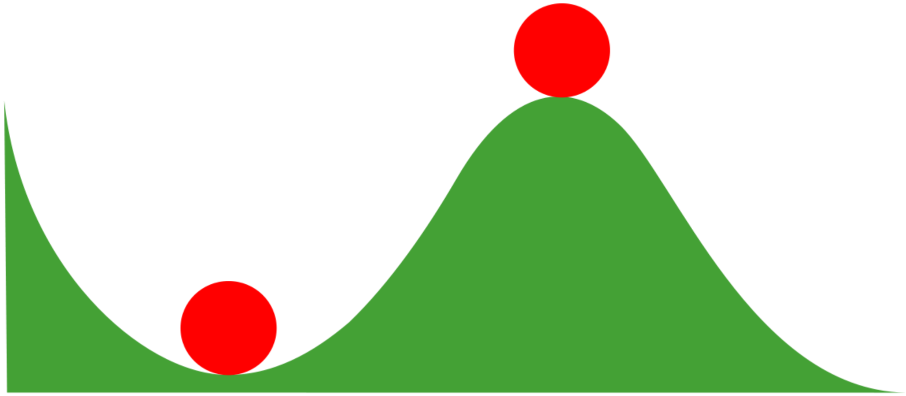
- This equilibrium is stable if A is Hurwitz

Equilibria and stability

The system is unstable if the matrix A

1. has eigenvalues in the **right-half Gauss plane**
2. has eigenvalues with **0 real-part** and multiplicity > 1

(if eigenvalues with 0 real part are present with multiplicity one, the system is said to be **marginally stable**)



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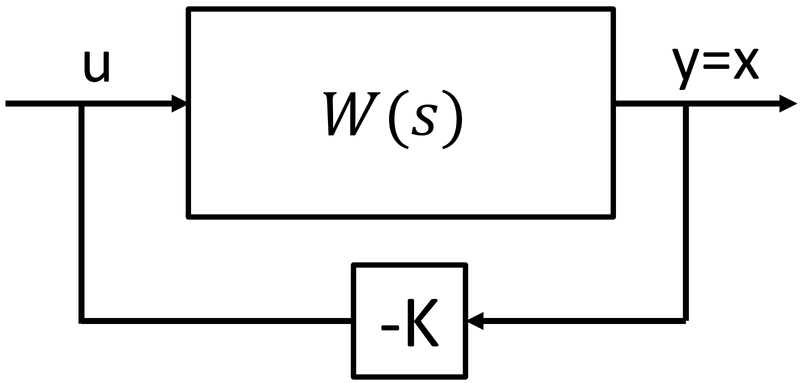
Example: state feedback

- A simple example of feedback control is as follows (**state feedback**)

$$\dot{\boldsymbol{x}} = A\boldsymbol{x}(t) + Bu(t), \qquad x(t_0) = x_0$$

$$y = \boldsymbol{x}(t)$$

$$\boldsymbol{u} = -\boldsymbol{K}\boldsymbol{x}$$



Example: state feedback

- Substituting the expression of u(t):

$$\dot{\boldsymbol{x}} = (A - BK)\boldsymbol{x}(t)$$

- With this choice, the original system is recast into an autonomous system with dynamical matrix $\boldsymbol{A - BK}$

- The system will evolve with the modes associated to $A - BK$: if these can be chosen arbitrarily (i.e. the state is *measurable* and the system is *controllable*…), we are free to assign a desired dynamic behaviour to the system!

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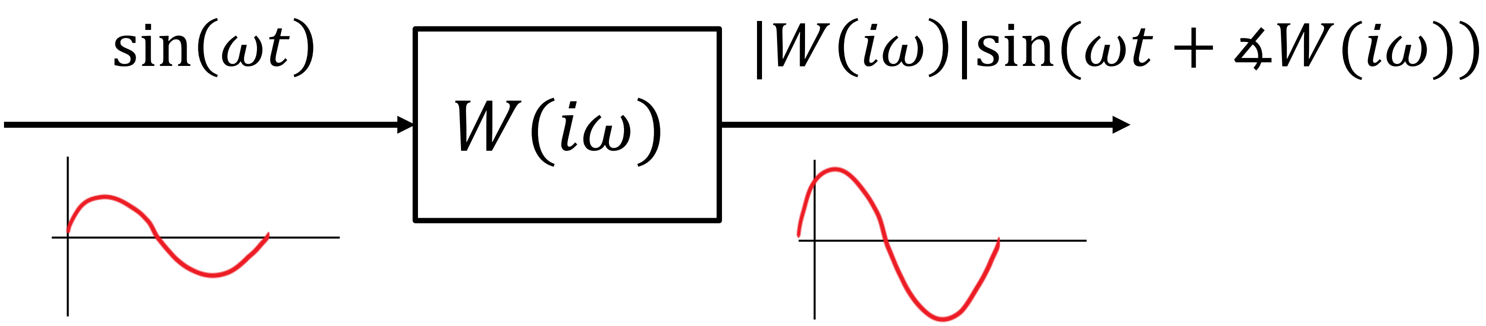
Frequency response

- **Frequency domain analysis** is a tool widely used in the analysis and control of dynamical systems
- The **harmonic response function** can be obtained by applying the **Fourier transform** to the system
- For <u>stable</u>* LTI systems this is equivalent to evaluate the tf along the imaginary axis $s = i\omega$

* Stable systems have convergence abscissa < 0

Frequency response

- A LTI system under a sinusoidal input will produce a **sinusoidal output**
- **Module and phase** of the output sinusoid are related to the harmonic response function



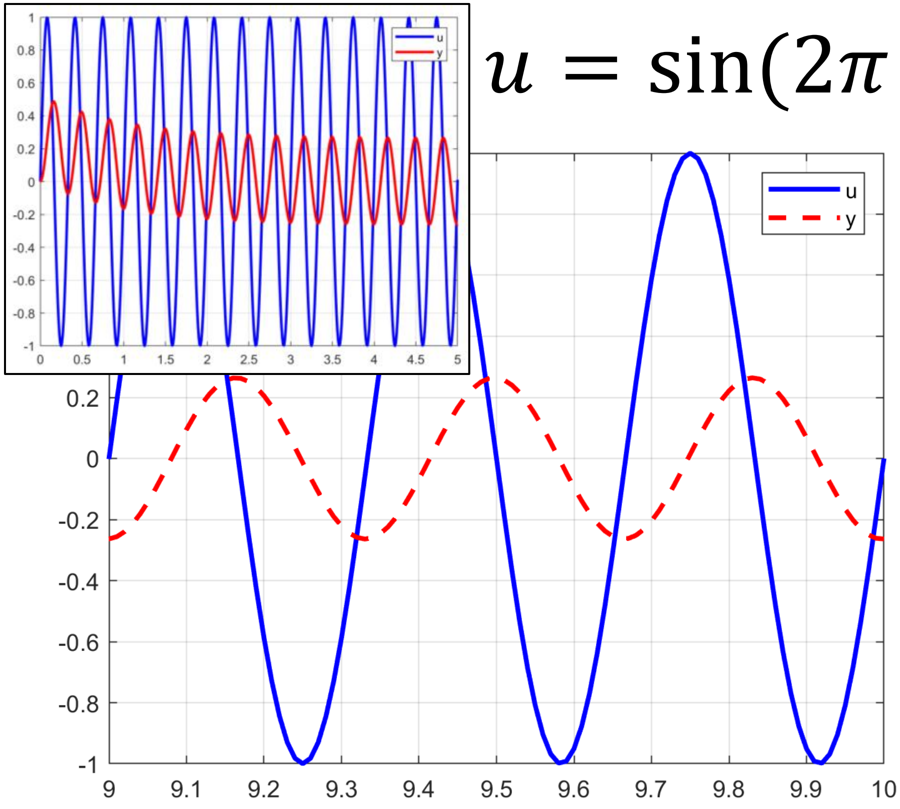
Frequency response

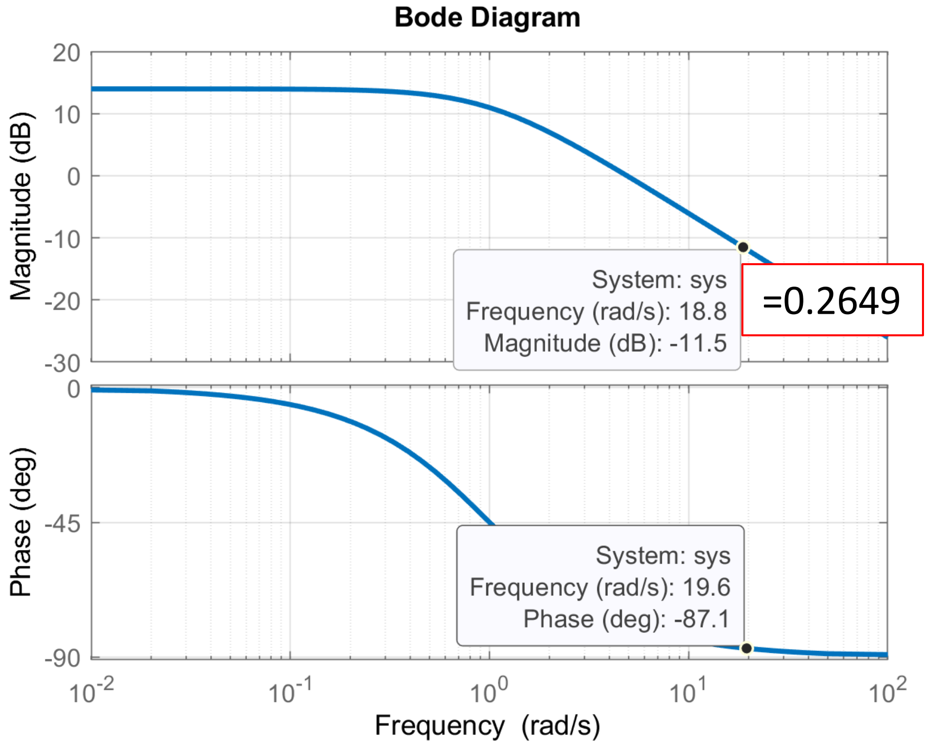
- It can be shown that feeding a sinusoid $u(t) = \sin(\omega_0 t)$ to a LTI system, it reaches sinusoidal regime, where:
    - the output is shifted by $\measuredangle \boldsymbol{P}(\boldsymbol{\omega_0})$
    - it is scaled by $|\boldsymbol{P}(\boldsymbol{\omega_0})|$
- NOTE: $P(i\omega) = -P^*(i\omega)$ → phase is odd, module is even
  (…of course - can you see why?)

Frequency response

- example:

$$\dot{x} = -x + 5u, \qquad y = x$$
$$u = \sin(2\pi \cdot 3t)$$



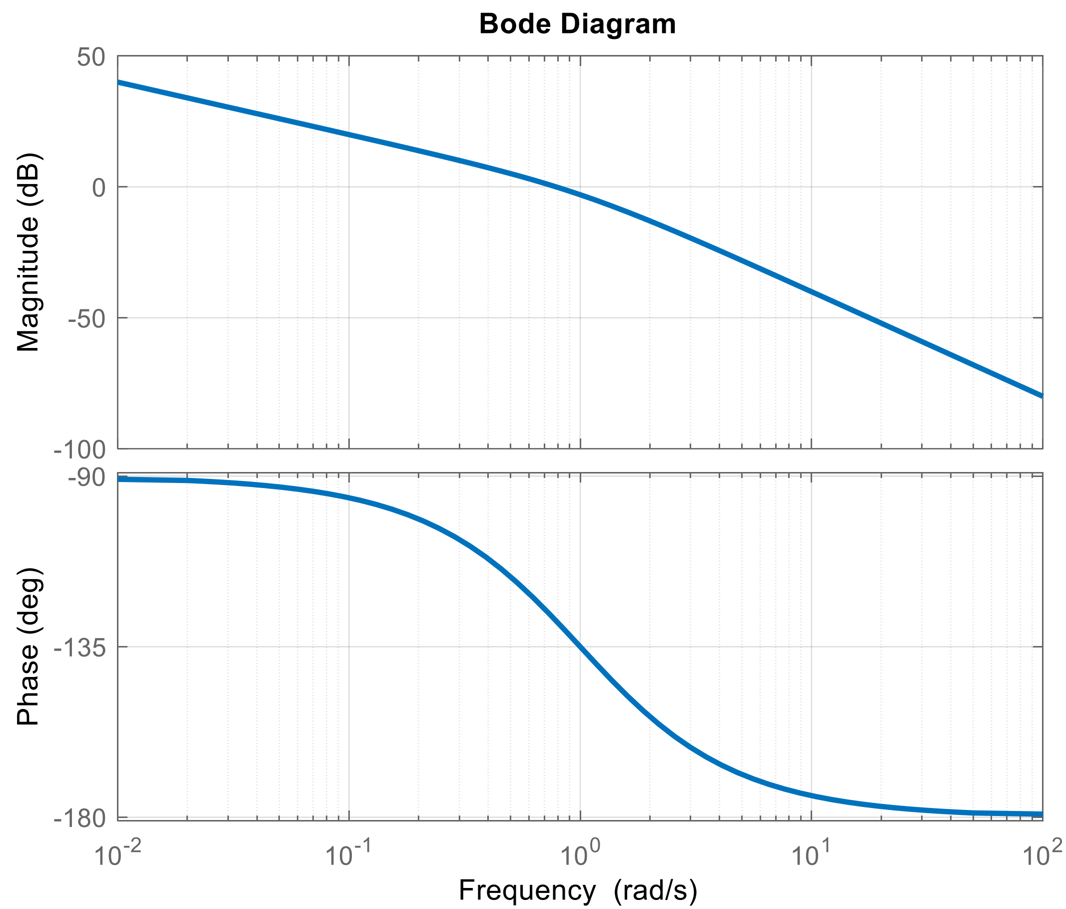


Graphical representation

- Some graphical representations are widely used in control engineering
- They focus on different aspects of the harmonic response

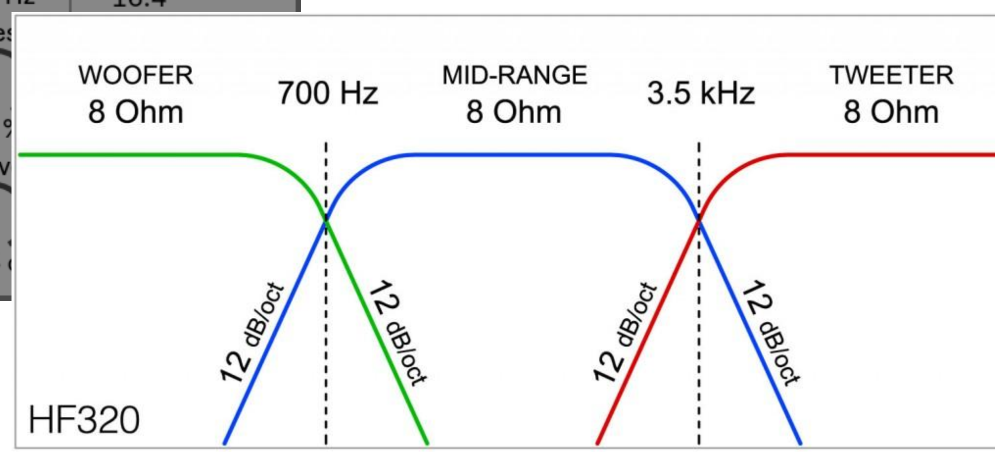
Bode diagram

- **Magnitude** and **phase** of $G(i\omega)$ separately
- wide frequency ranges
  → usually x axis in **logarithmic scale**
- Example:
  $$P(i\omega) = \frac{1}{s(s+1)}$$



Bode diagram

- Bode diagrams (magnitude in particular) are quite common also in other fields



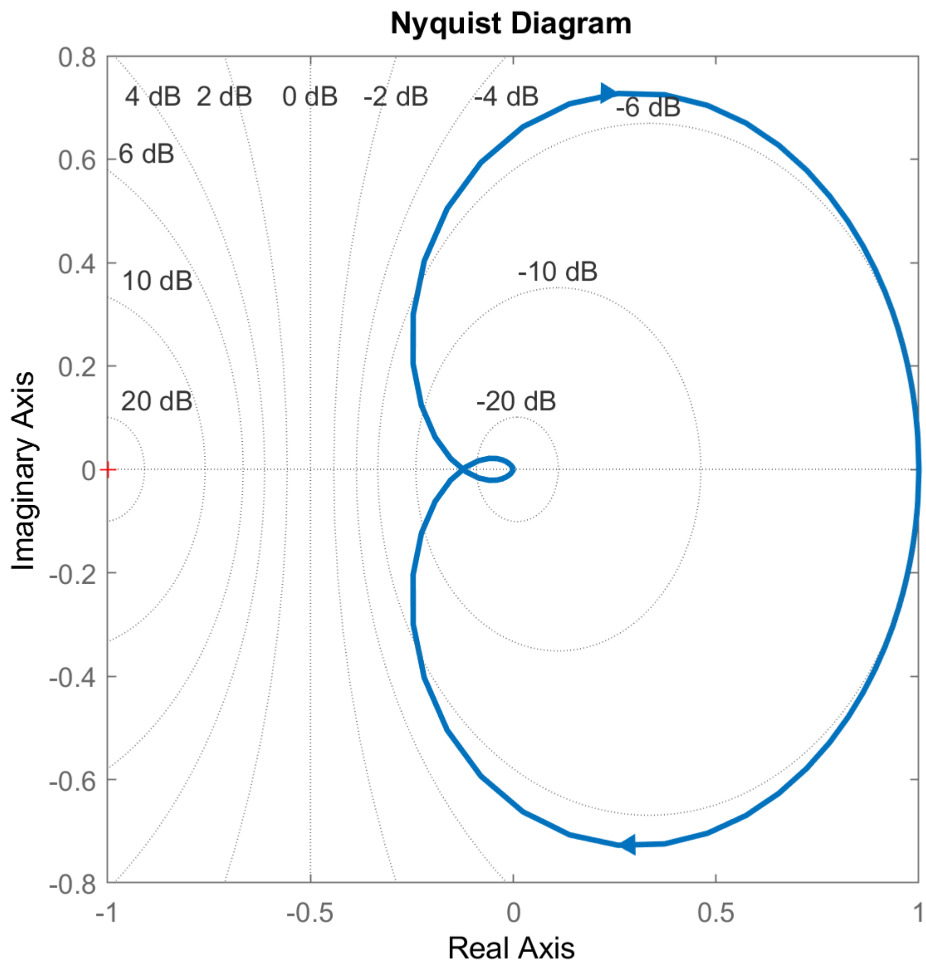
Nyquist diagram

- **Real and imaginary parts** of $G(i\omega)$ on the same graph

- Very useful to study stability

- Example:

$$G(i\omega) = \frac{1}{(s+1)^3}$$



Discrete-time systems

- In many cases, it may be necessary to discretize the dynamics of the system

- Typical example: **digital control systems**

$$\dot{x} = Ax(t) + Bu(t) \;\rightarrow\; x(k+1) = \tilde{A}x(k) + \tilde{B}u(k)$$

Z-transform

- In this case the Z-transform can be used

$$F(s) = \int_0^{+\infty} e^{-st} f(t)dt$$

- represent sampled signal as

$$t = kT_s \Rightarrow f(kT_s) \approx \sum_0^{\infty} f(t)\delta(t - kT_s)$$

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Z-transform

- apply Laplace transform

$$F(s) = \int_0^{+\infty} e^{-st} \sum_0^{\infty} f(t)\delta(t - kT_s)\, dt$$
$$= \sum_0^{\infty} f(kT_s)\, e^{-skT_s}$$

- so it is natural to define

$$z = e^{sT_s} \Rightarrow F(z) = \sum_0^{\infty} f(kT_s)\, z^{-k}$$

Z-transform

The Z-transform inherits many properties

- Linearity
- Translation in time

$$Z[f(k-1)] = F(z)z^{-1}$$

  - sometimes written as

$$f(k-1) = z^{-1}f(k)$$

- Final value

$$\lim_{t\to\infty} f(t) = \lim_{z\to 1}(z-1)F(z)$$

Transfer function (discrete case)

- Similar to the Laplace transform

$$zX(z) - x_0 = AX(z) + BU(z)$$
$$Y(z) = CX(z) + DU(z)$$

$$\Downarrow$$

$$Y(z) = C(zI - A)^{-1}x_0 + \underbrace{\big(C(zI - A)^{-1}B + D\big)}_{\text{again a rational function}}U(z)$$

Hurwitz theorem (discrete case)

- For continuous systems

$$\dot{x} = \lambda x \rightarrow x(t) = e^{\lambda(t-t_0)} x_0$$

  – stability $\Leftrightarrow Re(\lambda) < 0$

- For discrete systems

$$x_{k+1} = \lambda x_k \rightarrow x_k = \lambda^k x_0$$

  – stability $\Leftrightarrow |\lambda| < 1$

Continuous to discrete

- To approximate a continuous system with a discrete one, we could just apply the substitution

$$z = e^{sT_s} \Rightarrow s = \frac{1}{t} \ln z$$

in the Laplace transform

- Unfortunately, this is not linear!

Continuous to discrete

- Different approximations are possible

$$sX(s) \leftrightarrow ?\, X(z)$$

- Forward/Explicit Euler (FE)

$$sX(s) \rightarrow \frac{x(k+1) - x(k)}{T_s} \rightarrow \frac{z-1}{T_s} X(z)$$

- Backward/implicit Euler (BE)

$$sX(s) \rightarrow \frac{x(k) - x(k-1)}{T_s} \rightarrow \frac{1-z^{-1}}{T_s} X(z)$$

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Continuous to discrete

- Trapezoids (or Crank-Nicholson, or Tustin)

$$y(t) = \int_0^{t_k} x(\tau)d\tau = y(k-1) + \int_{t_{k-1}}^{t_k} x(\tau)d\tau$$

$$\approx y(k-1) + \frac{x(k) + x(k-1)}{2} T_s$$

$$\rightarrow Y(z) = z^{-1}Y(z) + X(z)\left(1 + z^{-1}\right)\frac{T_s}{2}$$

$$\frac{X(s)}{s} \rightarrow \frac{1 - z^{-1}}{1 + z^{-1}} \frac{2}{T_s} X(z)$$